Francesco Del Medico

Buckling of laminated-composite cylindrical shells under axial compression

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Buckling of laminated-composite cylindrical shells under axial compression

Candidato:

Francesco Del Medico ....

Relatori:

Ing. Michele Biagi ....

Prof. Marco Beghini ....

Prof. Leonardo Bertini ....

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Ai miei genitori

Abstract

A deep investigation upon stability of laminated-composite cylindrical shells under axial compression is presented and analytical methods for shell design criteria are demonstrated and discussed.

In the first part the classical Donnell-type solution of the buckling problem of geometrically perfect cylindrical shells is presented; a comparison between the results found out with this solution and those obtained with Finite Elements Analysis in NASTRAN highlights the need of different approaches for orthotropic shells. New algorithms SOLBUC-1 and SOLBUC-2 are proposed and appear in good accordance with FEM results.

In the second part, an analysis of experimental results on a selected group of laminated-composite cylindrical shells puts in evidence the high sensibility of these structures to geometrical imperfections.

NASA approach currently used in industry is based on the introduction of "*knock-down factors*" that, infact, yield over-conservative Critical Buckling Loads; new design approaches are proposed leading to a good mark-up in comparison to NASA results.

In particular, an analytical solution of non-linear fourth-order partial differential stability equations considering axis-symmetrical imperfection is presented in new algorithm SOLDIF and leads to new Buckling Load "*knock-down factors*".

These new "*knockdown factors*" appear substantially improved and hence less conservative than the corresponding "*knockdown factors*" presently used in the industry.

Furthermore, the results show that the improved analytical-based knock-down factor presented always yields a safe estimate of the buckling load of the shells examined in this work.

Thanks

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a	membrane compliance
A	extensional stiffness matrix
B	bending-stretching coupling matrix
D	flexural matrix
$\tilde{D}$	modified bending stiffness defined by equation 2.10
e	eccentricity matrix
E_{11}, E_{22}	Young's moduli in the 1 and 2 directions
F	Airy's stress function defined by equation 2.16
K	skewedness parameter
$[K]$	FEM stiffness matrix
L	shell length
L_a, $L_{\tilde{D}}$, L_e	linear operators defined by equation 2.23
L_{NL}	non-linear operator defined by equation 2.26
m	number of axial half-waves
M	bending/twisting moments per unit length
M^*	external moment applied at boundaries
n	number of circumferential full waves
N	membrane forces per unit length
N_{Class}	Critical Buckling Load by Classical Solution
N_{cr}	Critical Buckling Load
N_p	number of plies of laminate
N_{SB1}	Critical Buckling Load by [SOLBUC-1]
N_{SB2}	Critical Buckling Load by [SOLBUC-2]

p_z	external lateral pressure
P^*	external forces applied at boundaries
$\overline{Q}$	reduced stiffness constants
R	shell radius
$[S]$	FEM stress-stiffening matrix
u	displacement in x (axial) direction
U_e	internal strain energy
v	displacement in y (circumferential) direction
w	displacement in z (radial) direction
w_ν	axial Poisson's effect
$\tilde{w}$	initial radial imperfection
W	work of external forces
T	total shell thickness
x,y	axial and circumferential coordinates on the middle surface of the shell, respectively (see Figure 2.1)
β	parameter defined by equation **??**
$\overline{\beta}$	parameter for characterization of axis-symmetric imperfection
ϵ_x, ϵ_y	total deformation
ϵ_{x0}, ϵ_{y0}, γ_{xy0}	deformation of the medium surface
η	parameter defined by equation **??**
Φ	Total Potential Energy
γ	Knock-down Factor for NASA
κ_x, κ_y	variation of curvature of the medium surface
λ, λ_c	critical eigenvalue
ν	reference Poisson Ratio
ξ	amplitude of imperfection
ξ_{sym}	amplitude of axis-symmetric imperfection

Superscripts:

(0)	pre-buckling solution
$\hat{}$	perturbation at bifurcation point

Chapter 1

Introduction

In this chapter a brief introduction to the problem of buckling in cylindrical shell-type structures is presented.

A brief introduction on the various types of equilibrium states of structures will be given in order to understand the behaviour of the *perfect* structural system in the passage from a stable configuration to an unstable one.

In the second part Koiter's theory [13] on *post-buckling* behaviour is briefly treated of to highlight the importance of geometrical imperfection in the determination of buckling-load of the *real* structure, that often results sensitively lower than that obtained for the perfect structure. In this sense, unfortunately, cylindrical shells' stability appears very sensitive to the influence of imperfection.

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1.1 Buckling of circular cylindrical shells

Thin-walled shells are frequently used as structural elements in such different applications like cooling towers, legs of offshore bore island, aircraft fuselages or as the main load carrying elements of aerospace launch vehicles.

The popularity of shells is due to the fact that they are very efficient load carrying structures; however, unfortunately, often they are prone to "catastrophic" elastic instabilities. Thus a through understanding of the stability behaviour of thin-walled shells is dutiful for all those who employ them.

1.2 Buckling of structural system

1.2.1 Basic concepts

A **structural system** is formed by a structure and the loads acting on it. There are two main properties that make a structure withstand loads:

- the constitutive material
- the geometric shape

Every structure is designed with a specific shape and it is expected that it should retain this shape during its service life.

Buckling is a process by which a structure cannot withstand loads with its original shape, so that it changes this shape in order to find a new equilibrium configuration. This is an undesired process (from the point of view of the engineer), and occurs at a well-defined value of the load. The consequences of buckling are basically geometric: there are large displacements in the structure, to such an extent that the shape changes and reduces its load-carrying capability. There may also be consequences for the material, in the sense that deflections may induce plasticity in some parts of the structure. Buckling is associated not just to a structure, but to the whole structural system.

To visualize a buckling process it is necessary to consider the load-deflection diagram, as shown in Figure 1.1. Here we plot the equilibrium states of the structure in terms of the load applied and the deflection obtained. Of course there are deflections in almost every point of the structure, so that it is necessary to choose a convenient point and follow the process by looking at the displacements of this specific point.

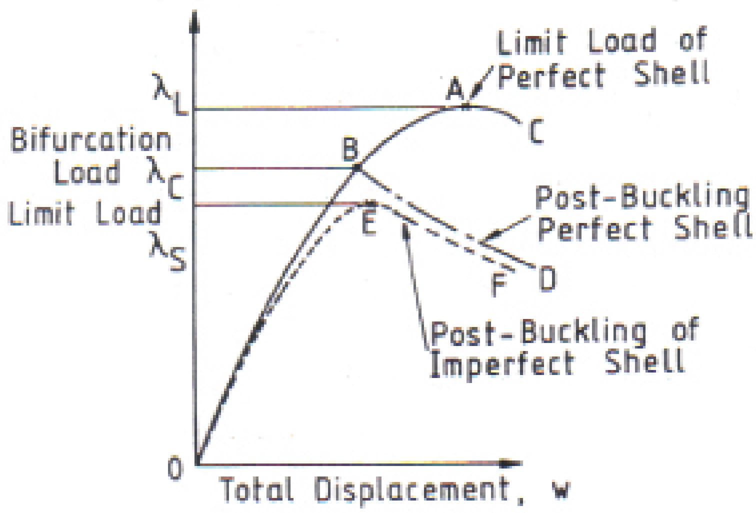

Figure 1.1: Deflection diagrams showing equilibrium paths limit point and bifurcation point.

1.2.2 Equilibrium of a mechanic system

From an analytical point of view, considering a physical behaviour of a mechanic system, whose evolution in time could be described with an equation like

$$\begin{cases} \dfrac{du}{dt} = \phi(\lambda, u) \qquad t > 0 \\ \\ u(0) = u_0 \end{cases}$$

where the parameter λ is a scalar real number, the stationary equilibrium states can be defined in the equation

$$\phi(\lambda, u) = 0$$

That leads to a solution ensemble $S = \{(\lambda, u); \phi(\lambda, u) = 0\}$ Applying the principle of virtual work (D'Alembert) it can be affirmed that the total potential energy Φ of a mechanic system loaded with conservative forces is stationary. Considering that the total potential energy Φ is given by

$$\Phi = U_e + W_{ext} \tag{1.1}$$

where

- U_e is the internal strain energy of the structure
- W_{ext} is the work made by external forces.

As a consequence, the ensemble S of the mechanical system equilibrium points is represented simply by solutions of the following equation

$$\partial\Phi = \partial U_{int} + \partial W_{ext} = 0 \tag{1.2}$$

If the solution ensemble S has not qualitative changes when the parameter λ varies in a specified interval, the equilibrium point is called regular; on the contrary, if a qualitative variation is detectable for an arbitrarily little variation of parameter λ, around a value λ_c, this value is considered a critical one and the equilibrium state is defined singular.

With reference to 1.2.3 these points indicate the presence of equilibrium bifurcation.

1.2.3 Stability of equilibrium points

At a bifurcation point, with an increment of structural forces, i.e. of parameter λ, the structure can follow both the initial primary equilibrium path or shifting on a secondary pattern, as clearly showed in Figure 1.1; indeed, both primary and secondary path represent solutions of the equilibrium equation 1.2 and there is not any evident reason to have a particular behaviour of the mechanical system. In fact, the theorem of Total Potential Energy allows to say that, at an equilibrium state, the displacements that realize the equilibrium conditions are such that potential energy is minimized.

With reference to 1.2, considering that first-derivative term is equal to zero at an equilibrium point, the presence of a minimum leads immediately to the following condition for a stable equilibrium state

$$\begin{cases} \Phi'(u,\lambda)=0 \\ \\ \Phi''(u,\lambda)>0 \end{cases} \Rightarrow \qquad \text{stable equilibrium} \tag{1.3}$$

For most real structural system, the secondary path represents the minimum of Total Potential Energy; so, for $\lambda > \lambda_c$ the primary pattern represents points of instable equilibrium.

1.2.4 Equilibrium paths and critical states

The sequence of equilibrium points in this diagram is known as an **equilibrium path**. The equilibrium path emerging from the unloaded configuration is called the fundamental or primary path, also the **prebuckling path**. This path may be linear (or almost linear) or may be nonlinear.

The load level at which there is a change in the shape is called **buckling load**, and the emerging geometry is called the buckling mode. There are several ways in which this process may happen:

- in **snap buckling**, the fundamental path is nonlinear and reaches a maximum load, at which the tangent to the path is horizontal. This state is called limit point (Figure 1.1);

- in **bifurcation buckling**, the fundamental path may be linear and it crosses another equilibrium path, which was not present at the beginning of the loading process (Figure 1.1). The state at which both paths cross is called a bifurcation point. Both limit and bifurcation points are called critical points or critical states.

Buckling is associated to a property of the equilibrium states known as stability. A stable equilibrium state is one in which, if there is a small disturbance to the system at the same load level, then the system oscillates but returns to the original state after a while. If the system does not return to the original state and goes to a new state, perhaps far from the original one, then the original was an unstable equilibrium state. At a critical point the stability changes from stable to unstable. The process that occours following

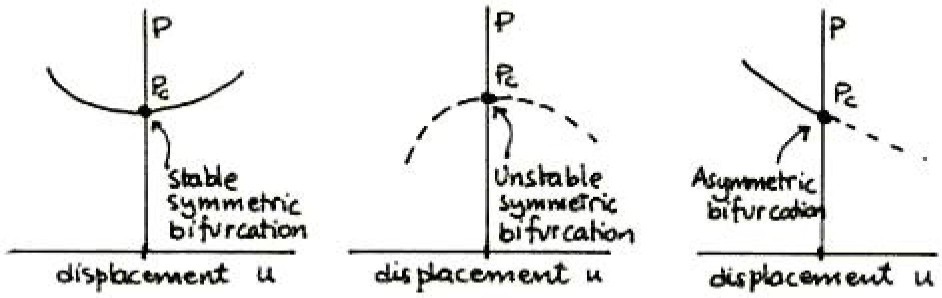

Figure 1.2: Three basic types of bifurcation for isolated modes.

buckling is called **post-buckling**.

There are structures with a load capacity in their post-buckling behavior, which can adjust to changes in shape and resist additional loads after buckling. Thus, there is a post-buckling equilibrium path, which may be stable. Other structures do not have stable post-buckling equilibrium states, so that the critical load is the maximum load of the structure.

In 1945, Koiter [13] showed that the critical states of bifurcation might be of the following types (see Figure 1.2):

- Stable symmetric bifurcation: the post-buckling path (also called secondary path) has a horizontal tangent at the critical point, and the path is stable, so that the structure can carry further load increments (Figure 1.2.a). This behavior is found in columns and plates;

- Unstable symmetric bifurcation: the post-buckling path has a horizontal tangent at the critical point, but the path is unstable, so that the structure cannot carry further load increments (Figure 1.2.b). This behavior is typically found in shells.

- Asymmetric bifurcation: the post-buckling path has a non-horizontal tangent at the critical point, and the path is stable on one side and unstable on the other, depending of the displacements (Figure 1.2.c).

So that the structure can carry further load increments only on the stable branch. This behavior is found in frames.

The type of behavior of Figure 1.2 occurs whenever there is an isolated critical state, also called distinct critical point. This means that the critical state is associated to just one buckling mode. There are also cases in which

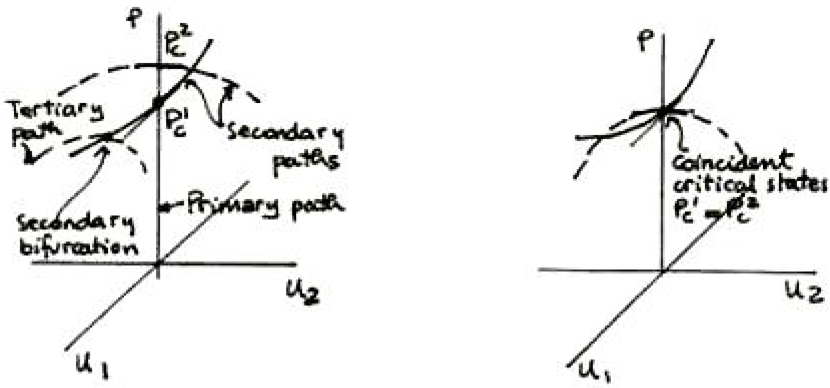

Figure 1.3: (a) Almost-coincident and (b) coincident critical states: two or more critical modes are associated to the same critical load.

there are two modes associated to the same critical load, and this is known as a coincident critical state, also known as a compound critical point. This situation is shown in Figure 1.3.

The case of almost coincident critical loads is presented in Figure 1.3, while coincident critical loads are shown in Figure 1.3(b). There are two reasons explaining how two or more modes can be coincident (or almost coincident):

- due to the selection of some design parameters, two modes that may otherwise take different values of critical load, could result coincident. In this case coincidence is the exception, not the rule;
- due to a problem of the structure and the loading considered. For example, cylindrical shells under axial load (as shown better in the following chapters) or spherical shells under uniform external pressure (two common geometries in the design of tanks) develop many coincident modes for the lowest critical state. Here it does not matter how we

design the shell, it will have coincident critical states, and coincidence is the rule, not the exception.

Two or more coincident (or almost coincident) critical states may have mode-coupling to form a new equilibrium path, different from the isolated equilibrium paths. For example, in Figure 1.3, the coupling of two modes produces a new secondary bifurcation state and a new tertiary equilibrium path. Not all coincident states couple, and there are several ways in which they may couple.

In many cases at the critical state the structure has a critical mode, and as the structure follows the postcritical equilibrium path the mode of deflections change. This is called **mode-jumping**.

1.2.5 Influence of imperfections

Many structural systems are sensitive to the influence of small imperfections. Examples of imperfections are geometric deviations of the perfect shape, eccentricities in the loads, local changes in the properties, and others. An imperfection is usually characterized by its variation in space and its amplitude ξ. An imperfection destroys a bifurcation point, and a new equilibrium path is obtained for each imperfection amplitude ξ. As the amplitude of the imperfection increases, the paths deviate more from the path of the perfect system. This is shown in Figure 1.4.

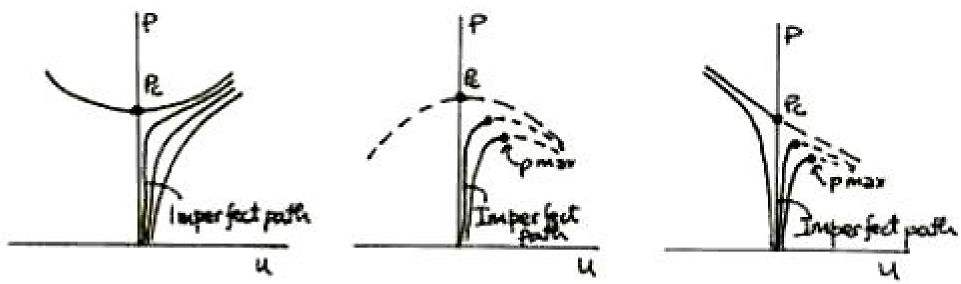

Figure 1.4: Influence of imperfections on bifurcation behavior of structural systems.

Structural systems that display stable symmetric bifurcations have a nonlinear path due to imperfections, and the bifurcation point is not reached (Figure 1.4.a). Systems with unstable symmetric bifurcation in the perfect configuration, when an imperfection is included have a nonlinear path with a maximum in the load, after which the path descends (Figure 1.4.b). Thus, the maximum load that the system can reach depends on the amplitude of the imperfection, and is lower than what would be computed using the perfect geometry.

Finally, systems with asymmetric bifurcation have a maximum load on the unstable branch, leading to a maximum load (Figure 1.4.c).

A typical plot is made showing the maximum load versus the amplitude of the imperfection; that is known as an imperfection-sensitivity plot. An example is shown in Figure 1.5. Some structures have a loss of buckling-carrying capacity of 50 percent or more, including cylinders under axial load and spheres under pressure; they have high imperfection sensitivity. Other structures have moderate sensitivity, like cylinders under lateral pressure, which have a loss of about 20-30 percent. Finally, there are structures with small sensitivity, like plates under in-plane loads. Problems with coincident

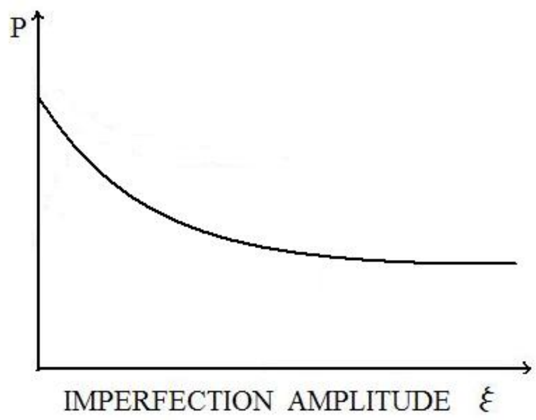

Figure 1.5: Imperfection sensitivity plot showing how the maximum load decreases with the amplitude of an imperfection.

(or almost coincident) critical states that have mode coupling may display high imperfection-sensitivity. This occurs in the cylinder and the sphere. In other cases (for example, an I-column under compression) there is mode-coupling but the imperfection-sensitivity is moderate.

1.2.6 Plastic buckling

The material properties during the buckling process are very important because they determine the process of loosing stability of the structure and the type of buckling and post-buckling behaviour.

In particular, three different processes can be experimentally observed with strict dependence on material properties:

- Elastic buckling: it is a process that initiates at the critical states with elastic material properties. Thus, instability occurs before plasticity: when the structure reaches plastic deformations it already experienced buckling. This occurs in most thin-walled shells, like tanks;

- Plastic buckling: it is a process that initiates with plastic deformations. Thus, plasticity occurs before instability: when the structure reaches a buckling load it already had plastic deformations. This occurs in thick shells;

- Elasto-plastic buckling: this occurs when plasticity and instability can be experienced almost at the same load level. This occurs in moderately thin shells.

1.3 Approaches to evaluate buckling

At present most structures are analyzed using a finite element model, and more specifically, a commercial computer package is employed like NASTRAN, ABAQUS, ALGOR, ADINA, ANSYS and others. As it is shown in the following analysis, there are basically three ways in which buckling-load may be evaluated using a finite element program:

- Bifurcation analysis: the program performs first a static analysis of a linear fundamental equilibrium path, and then computes the eigenvalues and eigenvectors of the system using the stiffness and the load-geometry matrices. The results are the buckling load and the buckling mode. No information is provided regarding the post-buckling path or sensitivity analysis;

- Nonlinear analysis: a step-by-step analysis is performed considering an initial imperfection and geometrical and material nonlinearity. Only limit points can be detected, and bifurcations are generally not taken into account. Anyway the software is able to perform a non-linear bifurcation analysis evaluating the value of the load at which a singularity is found out in the stiffness-matrix. Sometimes the program may fail to detect a bifurcation state. The results are a list of load and displacement configurations;

- Initial post-critical analysis: Koiter developed a theory in which the stability of the critical state provides information about the post-critical states close to the critical point. A perturbation analysis is performed to compute the initial post-critical secondary path. Commercial computer programs do not have this capability, and it has been incorporated into many special purpose finite element programs for shells.

Chapter 2

Stability-theory of thin walled cylindrical shells

Before carrying out meaningful comparison with experiments on cylinders' buckling, it is first of all necessary to understand the basic phenomena of structural instability.

In this chapter the theoretical concepts of the basic instability phenomena are presented; in the first part the hypothesis on deformation of thin-walled cylindrical shells are introduced taking in account the model proposed by Donnell in [5] for isotropic shells.

In the second part Total Potential Energy criterion is introduced in order to approach a stability problem and find out the Stability Equations, that are essential for an analytic resolution of any buckling analysis.

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2.1 Definition of thin-walled cylindrical shell

A thin-walled shell is a structure where the thickness T is much lower than the other two representative dimensions (typically $\dfrac{R}{T} > 100$); this property allows to transform a typical three-dimensional problem into a simpler two-dimensional one, i.e. the deformation field U of the structure can be referred uniquely to the "medium" surface, describing anyway with good approximation the displacements of out-of-plane points of the structure.

2.2 Stability-Theory for laminated-composite shells

The reduction of thin-walled shell into two dimensions leaded to some different theories that differ each other only in what regards the displacements considered in the evaluation of deformation and the terms of strength tensor; anyway, in this work only the Donnell's theory, elaborated until 1933 [5] on isotropic cylindrical shells is taken in account because their relative simplicity makes them ideally suited for rapid approximated analytical developments.

2.2.1 Hypothesis of behaviour

Donnell's theory can be applied in the case of little deformations of a cylindrical shell of radius R, length L and thickness T and the displacement fields $\{u, v, w\}$ refers to the coordinate system $\{x, y, z\}$ as showed in Figure 2.1.

The most important hypothesis of Donnell's theory are mainly:

1. shell-thickness T is negligible compared to length L and radius R, so that
$$T/R \ll 1 \qquad T/L \ll 1$$
2. the deformation are little, i.e. $\epsilon \ll 1$, and the application of Hooke's law is allowed;
3. the straight lines normal to medium undeformed surface remain straight and normal to the medium deformed surface without changing their length;

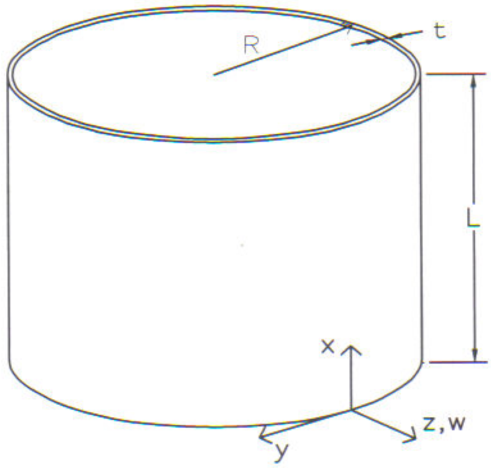

Figure 2.1: Cylinder geometry and main directions.

4. the strength tensor term relative to the direction normal to medium surface (z-z) is negligible if compared to terms in the other two directions, i.e.

$$\sigma_{zz} \ll \sigma_{yy} \qquad \sigma_{zz} \ll \sigma_{xx}$$

5. the displacements u e v are inappreciable and the displacement w is comparable to the thickness T of the shell, i.e.

$$|u| \ll T \qquad |V| \ll T \qquad |w| = o(T)$$

6. the derivative terms of w are negligible, but their squares and products are comparable to the absolute value of the deformations, i.e.

$$\left\{ \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y} \right\} \ll 1, \quad \left\{ \left(\frac{\partial w}{\partial x} \right)^2, \left(\frac{\partial w}{\partial y} \right)^2, \left| \frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y} \right| \right\} = o(\epsilon)$$

7. the changes in curvature have little values and the influence of u e v are negligeable; as a consequence, these terms can be represented only by linear function of the radial displacement w.

The hypothesis 3 and 4 represent the well known **Kirchoff-Love** hypothesis, while assumptions reported between 5 e 7 highlight the fact that the displacement is mainly radial.

Otherwise, the utilization of these assumption was often criticized, but it appears reasonable at all for a first evaluation of buckling load of a cylindrical structure.

2.2.2 Relation Deformation-Displacement

Based on these hypothesis, the relations between displacement and deformation can be expressed in the following form

$$\epsilon_x = \epsilon_{x0} + z \cdot \kappa_x \qquad \text{with} \qquad \epsilon_{x0} = \frac{\partial w}{\partial y} + \frac{1}{2} \cdot \left(\frac{\partial w}{\partial x}\right)^2 \qquad \kappa_x = -\frac{\partial^2 w}{\partial x^2}$$

$$\epsilon_y = \epsilon_{y0} + z \cdot \kappa_y \qquad \text{with} \qquad \epsilon_{y0} = \frac{\partial v}{\partial y} - \frac{w}{R} + \frac{1}{2} \cdot \left(\frac{\partial w}{\partial y}\right)^2 \qquad \kappa_y = -\frac{\partial^2 w}{\partial y^2}$$

$$\gamma_{xy} = \gamma_{xy0} + z \cdot \kappa_{xy} \qquad \text{with} \qquad \gamma_{xy0} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y} \qquad \kappa_{xy} = -2\frac{\partial^2 w}{\partial x \partial y}$$

$$(2.1)$$

where, with reference to List of Symbols, ϵ_{x0}, ϵ_{y0} and γ_{xy0} represent the deformations of the medium surface and κ_x, κ_y and κ_{xy} are the variations in curvature of the medium surface.

2.2.3 Constitutive law

For a laminated-composite material a linear relation between the increments of stress resultants and strain or change of curvature can be found; in particular

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} A & B \\ \text{sym} & D \end{bmatrix} \begin{bmatrix} \epsilon \\ \kappa \end{bmatrix} \tag{2.2}$$

where

$$\mathbf{N} = \begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} \tag{2.3}$$

$$\mathbf{M} = \begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} \tag{2.4}$$

with clear meaning of the terms.

The in-plane stiffness matrix A, the extension-bending coupling matrix B and the bending stiffness matrix D have to be determined from the classical lamination theory as presented in many textbooks, for instance in [4]. In a single ply the stress-strain relation is

$$\sigma_k = \overline{Q}_k \epsilon_k$$

where stress and strain are related to the coordinate system (x, y) and the coefficients $\overline{Q}_k$ are called the *reduced stiffness constants* in a plane stress state. The matrices A, B and D of a laminate consisting of N_p layers are defined as

$$A = \sum_{k=1}^{N_p} \overline{Q}_k (z_k - z_{k-1})$$

$$B = \frac{1}{2} \sum_{k=1}^{N_p} \overline{Q}_k (z_k^2 - z_{k-1}^2)$$

$$D = \frac{1}{3} \sum_{k=1}^{N_p} \overline{Q}_k (z_k^3 - z_{k-1}^3)$$

and the well-known Constitutive Equation is defined as

$$\begin{bmatrix} \mathbf{N} \\ - \\ \mathbf{M} \end{bmatrix} = \begin{bmatrix} A & | & B \\ - & - & - \\ B & | & D \end{bmatrix} \begin{bmatrix} \epsilon \\ - \\ \kappa \end{bmatrix} \tag{2.5}$$

The first of the two matrix equations 2.2 is solved for the strain at the reference surface and the result is substituted in the second equation to obtain a semi-inverted version of the constitutive law,

$$\epsilon = A^{-1}(N - B\kappa) \tag{2.6}$$

$$M = B^T A^{-1} N + (D - B^T A^{-1} B)\kappa \tag{2.7}$$

After the introduction of

- the membrane compliance
$$a = A^{-1} \tag{2.8}$$
- the eccentricity matrix
$$e = A^{-1}B = aB \tag{2.9}$$
(which is non-symmetric, i.e. $e^T \neq e$)
- the modified bending stiffness
$$\tilde{D} = D - B^T A^{-1} B \tag{2.10}$$

the constitutive law can be written in a more concise way as

$$\epsilon = aN - e\kappa, \qquad M = e^T N + \widetilde{D}\kappa \tag{2.11}$$

2.2.4 Total Potential Energy

Neglecting transverse-shear flexibility [1](Kirkhoff-Love-type assumption), the deformation-energy of the laminated-composite shell is given by

$$\begin{aligned} U_e &= \frac{1}{2}\int_0^L \int_0^{2\pi R} \int_{-T/2}^{T/2} (\sigma_x \epsilon_x + \sigma_y \epsilon_y + \sigma_{xy}\gamma_{xy}) \cdot dxdydz \\ &= \frac{1}{2T}\int_0^L \int_0^{2\pi R} \int_{-T/2}^{T/2} (N_x \epsilon_x + N_y \epsilon_y + N_{xy}\gamma_{xy}) \cdot dxdydz \end{aligned} \tag{2.12}$$

[1]This assumption is valid as to laminated-composite shell, but it is not allowable if sandwich-type shells are considered

On the other side, the work of external forces shall be written as follows:

$$W = -\int_0^L \int_0^{2\pi R} (p_z w) \cdot dxdy - \int_0^{2\pi R} \left[P_x^* u + P_y^* v + P_z^* w - M_x^* \frac{\partial w}{\partial x} \right]_{x=0}^{x=L} \cdot dy \tag{2.13}$$

where p_z represents the external pressure on cylindrical shell and P_x^*,P_y^*,P_z^* and M_x^* are the external forces and momentum applied at shell iss boundaries.[2]

With reference to 1.1, the Total Potential Energy can be obtained with the sum of internal deformation and work of external forces, i.e.

$$\Phi = U_e + W.$$

2.2.5 Compatibility equation

Between the components of strain the following compatibility condition must hold in order to have a strain field coherent with the hypothesis and the geometric properties of the shell:

$$\epsilon_{x,yy} + \epsilon_{y,xx} - \gamma_{xy,xy} = w_{,xy}^2 - w_{,xx} w_{yy} - \frac{1}{R} w_{,xx} \tag{2.14}$$

2.2.6 Equilibrium equations

Starting from Minimum Potential Energy Criterion showed in 1.2.3 and using Calculus of Variations, the Criterion for Equilibrium in formula 1.3 yields the following Euler Equations

[2] In case of pure axial compression only P_x is acting

$$\begin{cases} N_{x,x} + N_{xy,y} = 0 \\ \\ N_{xy,x} + N_{y,y} = 0 \\ \\ M_{x,xx} + 2M_{xy,xy} + M_{y,yy} \\ \qquad + (N_x \cdot w_{,x} + N_{xy} \cdot w_{,y})_{,x} + (N_{xy} \cdot w_{,x} + N_y \cdot w_{,y})_{,y} \\ \qquad\qquad - \dfrac{1}{R} N_y + p_z = 0 \end{cases} \tag{2.15}$$

A more useful form of equations 2.15 is derived by substitution of a new function F, called **Airy's Function**, such that

$$\begin{cases} N_x = F_{,yy} \\ \\ N_y = F_{,xx} \\ \\ N_{xy} = -F_{xy} \end{cases} \tag{2.16}$$

The introduction of this F-function does not convey any physical meaning, but only leads to great computational advantages in the following.

With this choice, the first two equations in 2.15 appear easily verified; the strain field resulting from the stress function must satisfy the compatibility condition showed in 2.14.

With reference to 2.11, the following expressions of M_x, M_y, M_{xy} can be used in the third equation in 2.15:

$$\begin{cases} M_x = -\tilde{D}_{11} w_{,xx} - \tilde{D}_{12} w_{,yy} - 2\tilde{D}_{13} w_{,xy} + e_{11} N_x + e_{21} N_y + e_{31} N_{xy} \\ \\ M_y = -\tilde{D}_{12} w_{,xx} - \tilde{D}_{22} w_{,yy} - 2\tilde{D}_{23} w_{,xy} + e_{12} N_x + e_{22} N_y + e_{32} N_{xy} \\ \\ M_{xy} = -\tilde{D}_{13} w_{,xx} - \tilde{D}_{32} w_{,yy} - 2\tilde{D}_{33} w_{,xy} + e_{13} N_x + e_{23} N_y + e_{33} N_{xy} \end{cases} \tag{2.17}$$

By introduction of equations 2.16 in 2.15, the first equilibrium equation is obtained as follows:

$$\begin{aligned}
\tilde{D}_{11}w_{,xxxx} + \tilde{D}_{22}w_{,yyyy} + 4\tilde{D}_{13}w_{,xxxy} + 4\tilde{D}_{23}w_{,yyyx} \\
+ (2\tilde{D}_{12} + 4\tilde{D}_{33})w_{,xxyy} + e_{21}F_{xxxx} + e_{12}F_{yyyy} \\
+ (e_{11} + e_{22} - 2e_{33})F_{xxyy} + (2e_{23} - e_{31})F_{xxxy} \\
+ (2e_{13} - e_{32})F_{xyyy} = F_{,yy}w_{,xx} + F_{,xx}w_{,yy} \\
- 2F_{,xy}w_{,xy} + p_z + \frac{1}{R}F_{,xx}
\end{aligned} \tag{2.18}$$

In the same way, with reference to 2.11, the following expressions of ϵ_x, ϵ_y and ϵ_{xy} can be used in compatibility equation 2.14

$$\begin{cases}
\epsilon_x = a_{11}N_x + a_{12}N_y + a_{13}N_{xy} + e_{11}w_{,xx} + e_{12}w_{,yy} + 2e_{13}w_{xy} \\
\epsilon_y = a_{12}N_x + a_{22}N_y + a_{32}N_{xy} + e_{22}w_{,xx} + e_{21}w_{,yy} + 2e_{32}w_{xy} \\
\gamma_{xy} = a_{13}N_x + a_{23}N_y + a_{33}N_{xy} + e_{23}w_{,xx} + e_{21}w_{,yy} + 2e_{32}w_{xy}
\end{cases} \tag{2.19}$$

As a consequence, the second stability equation has the following form

$$\begin{aligned}
e_{21}w_{xxxx} + e_{12}w_{yyyy} + (e_{11} + e_{22} - 2e_{33})w_{xxyy} \\
+ (2e_{23} - e_{31})w_{xxxy} + (2e_{13} - e_{32})w_{xyyy} \\
+ a_{11}F_{,yyyy} + a_{22}F_{,xxxx} + (2a_{12} + a_{33})F_{,xxyy} \\
- 2a_{13}F_{,xyyy} - 2a_{23}F_{,xxxy} \\
= w_{,xy}^2 - w_{,xx}w_{,yy} - \frac{1}{R}w_{,xx}
\end{aligned} \tag{2.20}$$

The two resulting governing partial differential equations can be put in the following form

$$L_{\tilde{D}}(w) + L_e(F) = \frac{1}{R}F_{,xx} + L_{NL}(F, w) + p_z \tag{2.21}$$

$$L_e(w) + L_a(F) = -\frac{1}{R}w_{,xx} - \frac{1}{2}L_{NL}(w, w) \tag{2.22}$$

where the linear operators are

$$\begin{aligned} L_{\tilde{D}}() = \tilde{D}_{11}()_{,xxxx} + \tilde{D}_{22}()_{,yyyy} + 4\tilde{D}_{13}()_{,xxxy} \\ + (2\tilde{D}_{12} + 4\tilde{D}_{33})()_{,xxyy} + 4\tilde{D}_{23}()_{,yyyx} \end{aligned} \tag{2.23}$$

$$\begin{aligned} L_e() = e_{21}()_{xxxx} + e_{12}()_{yyyy} + (e_{11} + e_{22} - 2e_{33})()_{xxyy} \\ + (2e_{23} - e_{31})()_{xxxy} + (2e_{13} - e_{32})()_{xyyy} \end{aligned} \tag{2.24}$$

$$\begin{aligned} L_a() = a_{11}()_{,yyyy} + a_{22}()_{,xxxx} + (2a_{12} + a_{33})()_{,xxyy} \\ - 2a_{13}()_{,xyyy} - 2a_{23}()_{,xxxy} \end{aligned} \tag{2.25}$$

and the nonlinear operator is

$$L_{NL}(S, P) = S_{,xx}P_{yy} - 2S_{,xy}P_{xy} + S_{,yy}P_{xx} \tag{2.26}$$

Commas in the subscripts denote repeated partial differentiation with respect to the independent variables following the comma.

It has to be noted that the two equations 2.21 and 2.22 only represent the shell's behaviour in a prebuckling equilibrium state, without any reference to bifurcation loads of the structure that cannot be evaluated in the simple solution of these differential equations.

2.3 Stability equations

The linearized stability equations for the determination of the critical load N_{cr} at the bifurcation point can be derived by the application of the

Adjacent Equilibrium Criterion. To investigate the existence of adjacent equilibrium configuration it is assumed that close to a bifurcation point one can represent the solution as

$$\begin{cases} w = w^{(0)} + \hat{w} \\ \\ F = F^{(0)} + \hat{F} \end{cases} \tag{2.27}$$

where $w^{(0)}$, $F^{(0)}$ represent the prebuckling solutions along the fundamental path and $\hat{w}$, $\hat{F}$ represent small perturbations at the bifurcation point. Direct substitution of these expressions into 2.21 and 2.22 and deletion of second order terms [3] yields a set of nonlinear governing equations for the prebuckling quantities $w^{(0)}$, $F^{(0)}$ in the form

$$L_{\tilde{D}}(w^{(0)}) + L_e(F^{(0)}) = \frac{1}{R}F^{(0)}_{,xx} + L_{NL}(F^{(0)}, w^{(0)}) + p_z \tag{2.28}$$

$$L_e(w^{(0)}) + L_a(F^{(0)}) = -\frac{1}{R}w^{(0)}_{,xx} - \frac{1}{2}L_{NL}(w^{(0)}, w^{(0)}) \tag{2.29}$$

and a set of linearized stability equations governing the perturbation quantities

$$L_{\tilde{D}}(\hat{w}) + L_e(\hat{F}) = \frac{1}{R}\hat{F}_{,xx} + L_{NL}(\hat{F}, w^{(0)}) + L_{NL}(F^{(0)}, \hat{w}) \tag{2.30}$$

$$L_e(\hat{w}) + L_a(\hat{F}) = -\frac{1}{R}\hat{w}_{,xx} - L_{NL}(\hat{w}, w^{(0)}) \tag{2.31}$$

Finally, in equation 2.28 and 2.29, 2.30 and 2.31 both the behaviour of a cylindrical shell before buckling bifurcation and at bifurcation load are described; in the following chapter, these general results are applied to the specific situation of a cylindrical shell under axial compression in order to find out the critical buckling load that is the target of this work.

[3] squares and products of the perturbation quantities

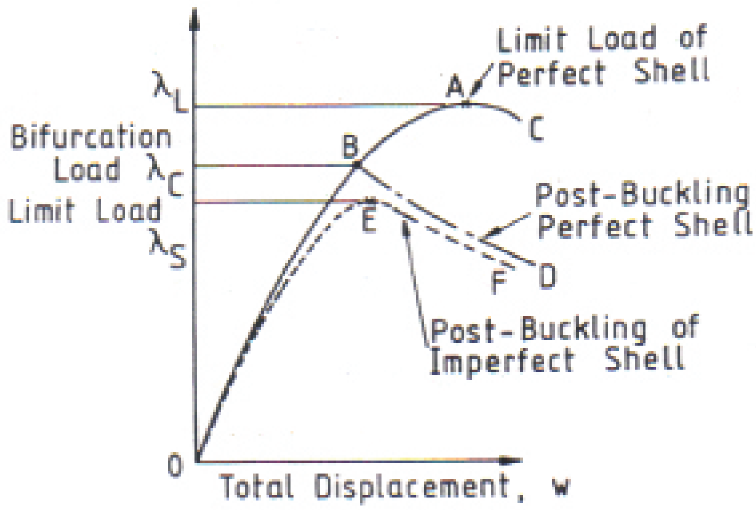

Figure 2.2: Deflection diagrams showing equilibrium paths limit point and bifurcation point.

Referring to Figure 2.2, if the calculation of the bifurcation load N_{cr} of the perfect shell is required, it is necessary finding the prebuckling solution along the path OB. This involves the solution of nonlinear equations 2.28 and 2.29 evaluated close to the load level corresponding to point B.

Then, these solutions are substituted into the linearized stability equations 2.30 and 2.31 yielding a nonlinear eigenvalue problem, where the lowest eigenvalue corresponds to the critical bifurcation load N_{cr}.

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